B. Sc. Part I (Hons) Trigonometry (1st Paper)

Gregory series

Named after James Gregory

1. If $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ then

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \dots \text{ to } \infty$$

Soln.

$$1 + i\tan\theta = 1 + i.\frac{\sin\theta}{\cos\theta} = \frac{\cos\theta + i\sin\theta}{\cos\theta}$$

$$\Rightarrow 1 + i\tan\theta = (\cos\theta + i\sin\theta)\sec\theta$$

$$\Rightarrow 1 + i\tan\theta = e^{i\theta}\sec\theta$$

$$\Rightarrow \log(1 + i\tan\theta) = \log(e^{i\theta}.\sec\theta)$$

$$\Rightarrow i\tan\theta - \frac{1}{2}(i\tan\theta)^2 + \frac{1}{3}(i\tan\theta)^3 - \frac{1}{4}(i\tan\theta)^4 + \dots$$

$$= \log e^{i\theta} + \log\sec\theta$$

[$\because$ θ lies between $-\frac{\pi}{4}$ to $\frac{\pi}{4}$, $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ can be applied.]

$$\Rightarrow i\tan\theta + \frac{1}{2}\tan^2\theta - \frac{1}{3}i\tan^3\theta - \frac{1}{4}\tan^4\theta$$

$$+ \frac{1}{5}i\tan^5\theta - \dots \text{ to } \infty = i\theta + \log\sec\theta$$

$\Rightarrow \log \sec\theta + i\theta$

$$= \left(\frac{1}{2}\tan^2\theta - \frac{1}{4}\tan^4\theta + \cdots\right)$$

$$+ i\left(\tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \cdots\right)$$

Equating ~~real and~~ imaginary parts from both sides, we get

$$\Rightarrow \boxed{\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \cdots \text{to } \infty} \quad (1)$$

This is called $\boxed{\text{GREGORY SERIES}}$.

Put $\tan\theta = x \Rightarrow \theta = \tan^{-1}x$

so, (1) becomes

$$\boxed{\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \text{to } \infty}$$

Put $x = 1$ in the above, we've

$$\tan^{-1}1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \text{to } \infty$$

$$\Rightarrow \boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \text{to } \infty}$$